2:35 - 4:15
1:35 - 2:15
tutorial starts next week

Chapter 1

MATHEMATICAL BACKGROUND

A, B, C matrices

column
$\vec{a}, \vec{b}, \vec{c}$ vectors.

This chapter discusses the mathematics that is necessary for the development of the theory of linear programming. We are particularly interested in the solutions of a system of linear equations.

In this notes, matrices are denoted by capital letters $A, B, C, \cdots$. Vectors are denoted by bold face smaller letters $\mathbf{a}, \mathbf{b}, \mathbf{c}, \cdots, \mathbf{x}, \mathbf{y}, \mathbf{z}$. Vectors are column vectors. The transpose of a matrix A is A^T. $\mathbf{0}$ is the zero vector and $\mathbf{1}$ is a column vector of all 1's.

1.1 Linear Equations and Linear Programming

In *linear programming* (LP), we usually want to optimize the *objective variable* z which is a linear function of the *decision variables* x_i $(i = 1, 2, \cdots, n)$ under some linear constraints on x_i. Symbolically, we have

Constrained optimization

linear programming

linear

$$
\begin{array}{ll}
\text{optimize} & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{subject to} & \left\{ \begin{array}{ccc} a_{11}x_1 + \cdots + a_{1n}x_n & \leq & b_1, \\ \vdots & & \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n & \leq & b_m, \end{array} \right. \\
\text{where} & x_i \geq 0, \quad i = 1, 2, \cdots, n.
\end{array}
$$

objective functional

function $\to \mathbb{R}$

c_i, a_{ij} are given

x_i unknown.

In matrix form, we have

$$
\begin{array}{ll}
\text{optimize} & z = \mathbf{c}^T\mathbf{x} \\
\text{subject to} & \left\{ \begin{array}{l} A\mathbf{x} \leq \mathbf{b} \\ \mathbf{x} \geq \mathbf{0}\,. \end{array} \right.
\end{array}
$$

$(c_1, c_2, \cdots, c_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$A = (a_{ij})$

This is called the *canonical form* of an LP problem. Since inequality constraints are difficult to handle, in actual computation, we usually consider LP problems of the form

$$
\begin{array}{ll}
\max & z = \mathbf{c}^T\mathbf{x} \\
\text{subject to} & \left\{ \begin{array}{l} A\mathbf{x} = \mathbf{b} \\ \mathbf{x} \geq \mathbf{0}\,. \end{array} \right.
\end{array}
$$

This is called the *standard form* of the LP problem. Notice that the constraints in the canonical form are transformed to a system of linear equations. The equation $z = \mathbf{c}^T\mathbf{x}$ is called the *objective* *function* and the set of all $\mathbf{x} \in \mathbb{R}^n$ that satisfy this equation is a *hyperplane* in $\mathbb{R}^n$. The rest of this chapter is devoted to the study of the solutions of systems of linear equations and the properties of hyperplanes.

1.2 Systems of Linear Equations and Their Solutions

Consider a system of m simultaneous linear equations in n unknown variables $x_1, \cdots, x_n$:

R C
$m \times n$
$n \times m$

m rows

$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1, \\ \vdots & & & & & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m. \end{array}\right.$$

n columns

$\vec{b} \in \mathbb{R}^m$

In matrix form, we have

$$A\mathbf{x} = \mathbf{b}. \qquad A \in \mathbb{R}^{m\times n} \quad \vec{x} \in \mathbb{R}^n \tag{1.1}$$

To solve this system of equations is to find the values of $x_1, x_2, \cdots, x_n$ that satisfy the equation. The corresponding vector $\mathbf{x} = [x_1, x_2, \cdots, x_n]^T$ will be called a *solution* to (1.1).

To find the solutions of (1.1), we construct the $\boxed{\textit{augmented matrix}}$ A_b of A that is defined by

$\vec{a}_1 \quad \vec{a}_2$

rank (A)

$$A_b = [A, \mathbf{b}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix} = [\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n, \mathbf{b}] \ ,$$

where $\mathbf{a}_i$ is the i-th column of A. For the solution of (1.1), there are two cases to consider.

(a) rank$(A) <$ rank(A_b).

Then $\mathbf{b}$ and the columns of A are linearly independent. Hence there are no x_i such that

$\begin{bmatrix} 1 & 2 & \vdots & 3 \\ 1 & 2 & \vdots & 4 \end{bmatrix}$
$A \quad \vec{b}$

here

hyper- $x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = [\vec{a}_1, \vec{a}_2 \cdots \vec{a}_n]\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \vec{b}$

$$\sum_{i=1}^{n} x_i \mathbf{a}_i = \mathbf{b}.$$

$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ rank$(A) = 1$, rank$(A_b) = 2$

In particular, the system $A\mathbf{x} = \mathbf{b}$ has no solutions. In that case, we call the system *inconsistent*. Notice that here we have rank$(\mathbf{b}) = 1$ and rank $(A_b) =$ rank $(A) + 1$.

(b) rank$(A) =$ rank$(A_b) = k$.

Then every column of A_b, in particular the vector $\mathbf{b}$, can be expressed as a linear combination of k linearly independent columns of A, i.e. there exist $x_{i_1}, x_{i_2}, \cdots x_{i_k}$ not all zero such that

$\begin{bmatrix} 1 & 2 & \vdots & 3 \\ 1 & 2 & \vdots & 3 \end{bmatrix}$
$A \quad \vec{b}$

$$A\vec{x} = [\vec{a}_1 \cdots \vec{a}_n]\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_{j=1}^{k} x_{i_j} \mathbf{a}_{i_j} = \mathbf{b}.$$

$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ rank(A) $=$ rank$(A_b) = 1$

Thus at least one solution exists in this case. We remark that if $m = n =$ rank(A), then the solution is also unique and $\mathbf{x} = A^{-1}\mathbf{b}$. However, in LP, we usually have rank$(A) = m < n$ and $A\mathbf{x} = \mathbf{b}$ usually has more than one solution.

1.3 Properties of Solutions of Systems of Linear Equations

Let us suppose that $A\mathbf{x} = \mathbf{b}$ has more than one solution, say $\mathbf{x}_1$ and $\mathbf{x}_2$ with $\mathbf{x}_1 \neq \mathbf{x}_2$. Then for any $\lambda \in [0, 1]$,

$$A[\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2] = \lambda A\mathbf{x} + (1-\lambda)A\mathbf{x}_2 = \lambda\mathbf{b} + (1-\lambda)\mathbf{b} = \mathbf{b}.$$

p1

$\max \quad 30x + 20y$ — decision variable

$$\begin{pmatrix} 1 & 1 \\ 40 & 60 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \le \begin{pmatrix} 80 \\ 2400 \\ 40 \end{pmatrix} \quad \left\{ \begin{aligned} x + y + s &\,\overset{=}{\le}\, 50 \\ 40x + 60y &\le 2400 \\ x + 2y &\le 40 \\ x, y &\ge 0 \end{aligned} \right.$$

surplus. $s \ge 0$

$$3\;\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 40 & 60 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ s \\ t \\ u \end{pmatrix} = \begin{pmatrix} 50 \\ 2400 \\ 40 \end{pmatrix} \qquad \begin{matrix} x, y, \\ s, t, u \ge 0 \end{matrix}$$

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$C^5_2 = \frac{5 \cdot 4}{2} = 10$

$x = y = 0$

$$\begin{pmatrix} x \\ y \\ s \\ t \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 50 \\ 2400 \\ 40 \end{pmatrix}$$ is a feasible corner pt

$t = u = 0$
$$\begin{pmatrix} 1 & 1 & 1 \\ 40 & 60 & 0 \\ 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ s \end{pmatrix} = \begin{pmatrix} 50 \\ 2400 \\ 40 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ s \end{pmatrix} = \begin{pmatrix} 120 \\ -40 \\ -30 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ s \\ t \\ u \end{pmatrix} = \begin{pmatrix} 120 \\ -40 \\ -30 \\ 0 \\ 0 \end{pmatrix}$$ is an infeasible corner pt

$y < 0 \quad s < 0$

$y = u = 0$

$$\begin{pmatrix} 1 & 1 & 0 \\ 40 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ s \\ t \end{pmatrix} = \begin{pmatrix} 50 \\ 2800 \\ 40 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ s \\ t \end{pmatrix} = \begin{pmatrix} 40 \\ 10 \\ 800 \end{pmatrix}$$

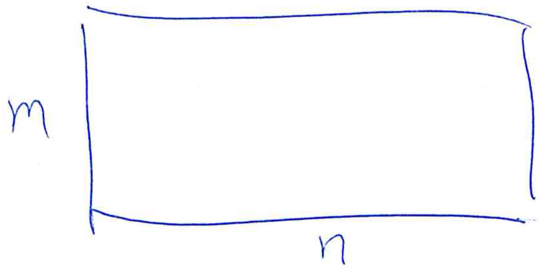
$\therefore \begin{pmatrix} x \\ y \\ s \\ t \\ u \end{pmatrix} = \begin{pmatrix} 40 \\ 0 \\ 10 \\ 800 \\ 0 \end{pmatrix}$ is a feasible corner pt

$x=10,\ y=10$

$$\begin{pmatrix} 1 & 1 \\ 40 & 60 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 10 \\ 10 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} s \\ t \\ u \end{pmatrix} = \begin{pmatrix} 50 \\ 2400 \\ 40 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} s \\ t \\ u \end{pmatrix} = \begin{pmatrix} 50 \\ 2400 \\ 40 \end{pmatrix} - \begin{pmatrix} 20 \\ 1000 \\ 30 \end{pmatrix} = \begin{pmatrix} 30 \\ 1400 \\ 10 \end{pmatrix}$$

$\therefore \begin{pmatrix} 10 \\ 10 \\ 30 \\ 1400 \\ 10 \end{pmatrix}$ is a feasible interior point.



$n - m$ variables $\equiv 0$

Solve for m variables.

$\boxed{\text{rank}}$

$=$ max # of linearly independent vectors inside the matrix.

$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ linear dependent

$2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ $\quad \frac{1}{2}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

rank $= 2$. $\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$ redundant. linear dependent

$\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$

$2\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$

$\boxed{\text{linear independent}}$

$=$ non redundant

rank $= 3$

$\det\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{pmatrix} \neq 0$ linearly independent

$[\vec{a}_1, \vec{a}_2, \vec{a}_3]$

$\exists\ \alpha_i$ not all zero.

s.t. $\alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2 + \alpha_3 \vec{a}_3 = \vec{0}$

$\alpha_1 = -1,\ \alpha_2 = 2,\ \alpha_3 = -1$

$\boxed{p \Rightarrow q} \iff \boxed{p \wedge \sim q = \text{contradiction}}$

proof by contradiction :

Assume $\sim q$ and p both true

then try to find a contribution